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A comment on the relationship of forces with stress tensors for fluids in a tube

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Abstract The force for fluid elements in a tube is discussed by considering a net force in an infinitesimal volume in incompressible fluids. An expression for it in terms of stress tensors is obtained in cylindrical coordinates, and it is critically compared with Zamir's results in a pulsating flow .

Keyword fluids ,　force,　stress tensors ,　cylindrical coordinates

1. Introduction

The Navier-Stokes equation is of fundamental importance to the fluid dynamics in various fields [1-3]. For instance, in biophysics, the pulsating flow in a tube was studied by Zamir [4], adopting cylindrical coordinates. To obtain an explicit expression for the equation of motion in curvilinear coordinates, one can apply a formal mathematical procedure to the vector Laplacian [5]. Alternatively, it is derived physically by considering the meaning of a net force per unit volume. These are equivalent, but the latter approach would be useful to investigate how each stress component on a surface effectively contributes to the force in a specified geometry.

In this note, we discuss a force per unit volume for incompressible fluids in a tube. Considering forces on a pair of surfaces in an infinitesimal volume, we obtain an expression of the net force in terms of stress tensors in cylindrical coordinates. The physical content of this is discussed in comparison with the Zamir's results [4].

2. Force per unit volume and stress tensors in cylindrical coordinates

In cartesian coordinates, the stress tensor σ_{ij} in incompressible viscous fluids [1] is given in terms of the viscosity coefficient η, pressure p and the fluid velocity $\boldsymbol{\upsilon}$ by

$$\sigma_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial \upsilon_j}{\partial x_i} + \frac{\partial \upsilon_i}{\partial x_j}\right) \qquad i, j = 1-3 \tag{1}$$

The quantities υ_1, υ_2, υ_3 stand for x, y, z components of the fluid velocity, respectively. With the use of the stress tensor (1) the force per unit volume is cast into the form

$$\boldsymbol{f} = -\nabla p + \eta \nabla^2 \boldsymbol{\upsilon} \tag{2}$$

The force $\boldsymbol{f}$ divided by the fluid density appears in the Navier-Stokes equation for the velocity, in incompressible fluids.

In the circular cylindrical coordinates, the parameters ℓ, φ, z are related to the cartesian coordinates (x, y, z) by

$$x = \ell\cos\varphi, \quad y = \ell\sin\varphi, \quad z = z \tag{3}$$

The unit base vectors $\boldsymbol{\varepsilon}_\ell$, $\boldsymbol{\varepsilon}_\varphi$, $\boldsymbol{\varepsilon}_z$ are defined by using the position vector $\boldsymbol{r}$ as

$$\boldsymbol{\varepsilon}_\ell = \frac{\partial \boldsymbol{r}}{\partial \ell}, \qquad \boldsymbol{\varepsilon}_\varphi = \frac{1}{\ell}\frac{\partial \boldsymbol{r}}{\partial \varphi}, \quad \boldsymbol{\varepsilon}_z = \frac{\partial \boldsymbol{r}}{\partial z} \tag{4}$$

Then, the velocity and the force are represented in the form

$$\begin{aligned} \boldsymbol{\upsilon} &= \upsilon_\ell \boldsymbol{\varepsilon}_\ell + \upsilon_\varphi \boldsymbol{\varepsilon}_\varphi + \upsilon_z \boldsymbol{\varepsilon}_z, \\ \boldsymbol{f} &= f_\ell \boldsymbol{\varepsilon}_\ell + f_\varphi \boldsymbol{\varepsilon}_\varphi + f_z \boldsymbol{\varepsilon}_z \end{aligned} \tag{5}$$

where υ_ℓ, υ_φ, υ_z and f_ℓ, f_φ, f_z are the physical components of υ and $\boldsymbol{f}$, respectively.

The components of stress tensors in cylindrical coordinates are $\sigma_{\ell\ell}$, $\sigma_{\ell\varphi} = \sigma_{\varphi\ell}$, $\sigma_{\ell z} = \sigma_{z\ell}$, $\sigma_{\varphi\varphi}$, $\sigma_{\varphi z} = \sigma_{z\varphi}$, σ_{zz} ; the quantity $\sigma_{\alpha\beta}$ represents the force per unit area along $\boldsymbol{\varepsilon}_\alpha$ on the infinitesimal surface perpendicular to the direction of $\boldsymbol{\varepsilon}_\beta$, with $\alpha, \beta = \ell, \varphi, z$. We consider a point specified by the parameters ℓ, φ, z, and also an infinitesimal region R around it by giving infinitesimal increments $\Delta\ell$, $\Delta\varphi$, Δz to these parameters. The volume of the region R is $\delta V = \ell\Delta\ell\Delta\varphi\Delta z$.

In the first place, we consider a pair of the infinitesimal surface defined by the fixed values of ℓ and $\ell + \Delta\ell$. At the point (ℓ, φ, z), the surface of the area $\ell\Delta\varphi\Delta z$ is perpendicular to the direction of $\boldsymbol{\varepsilon}_\ell$. Since $\sigma_{\ell\ell}$, $\sigma_{\varphi\ell}$, $\sigma_{z\ell}$ are the ℓ, φ, z components of the stress on the surface, respectively, the force acting on the surface corresponding to ℓ is given by

$$\boldsymbol{F}^{(\ell)}(\ell)=(\sigma_{\ell\ell}\boldsymbol{\varepsilon}_\ell+\sigma_{\varphi\ell}\boldsymbol{\varepsilon}_\varphi+\sigma_{z\ell}\boldsymbol{\varepsilon}_z)\ell\Delta\varphi\Delta z \qquad (6)$$

The superscript (ℓ) indicates the contribution from the surfaces perpendicular to $\boldsymbol{\varepsilon}_\ell$. Similarly, the force acting on the other surface corresponding to $\ell+\Delta\ell$ is

$$\boldsymbol{F}^{(\ell)}(\ell+\Delta\ell)=([\ell\sigma_{\ell\ell}]_{\ell+\Delta\ell}\boldsymbol{\varepsilon}_\ell+[\ell\sigma_{\varphi\ell}]_{\ell+\Delta\ell}\boldsymbol{\varepsilon}_\varphi+[\ell\sigma_{z\ell}]_{\ell+\Delta\ell}\boldsymbol{\varepsilon}_z)\Delta\varphi\Delta z \qquad (7)$$

where the notation such as $[\ell\sigma_{\ell\ell}]_{\ell+\Delta\ell}$ indicates that the value of $\ell\sigma_{\ell\ell}$ is evaluated at the point ($\ell+\Delta\ell$, φ , z) . The difference $\delta\boldsymbol{F}^{(\ell)}=\boldsymbol{F}^{(\ell)}(\ell+\Delta\ell)-\boldsymbol{F}^{(\ell)}(\ell)$ between these two forces is the net force for the fluid element in the region R. Dividing $\delta\boldsymbol{F}^{(\ell)}$ by δV , we obtain the force per unit volume $\boldsymbol{f}^{(\ell)}=\delta\boldsymbol{F}^{(\ell)}/\delta V$ as

$$\boldsymbol{f}^{(\ell)}=\frac{\partial}{\partial\ell}(\sigma_{\ell\ell}\ell)\boldsymbol{\varepsilon}_\ell+\frac{\partial}{\partial\ell}(\sigma_{\varphi\ell})\boldsymbol{\varepsilon}_\varphi+\frac{\partial}{\partial\ell}(\sigma_{z\ell})\boldsymbol{\varepsilon}_z \qquad (8)$$

Secondly, we consider a pair of the infinitesimal surfaces defined by the fixed values of φ and $\varphi+\Delta\varphi$. The force acting on the surface of the area $\Delta\ell\Delta z$ is given by $\boldsymbol{F}^{(\varphi)}(\varphi)=(\sigma_{\ell\varphi}\boldsymbol{\varepsilon}_\ell+\sigma_{\varphi\varphi}\boldsymbol{\varepsilon}_\varphi+\sigma_{z\varphi}\boldsymbol{\varepsilon}_z)\Delta\ell\Delta z$ just as in equation (7), omitting the subscript φ. But, for the force $\boldsymbol{F}^{(\varphi)}(\varphi+\Delta\varphi)$ acting on the surface corresponding to $\varphi+\Delta\varphi$, we have to take into account the fact that the unit vectors $\boldsymbol{\varepsilon}_\ell$ and $\boldsymbol{\varepsilon}_\varphi$ given by equation (4) themselves depend on the variable φ. As a consequence, it is appropriate to write $\vec{F}^{(\varphi)}(\varphi+\Delta\varphi)$ as

$$\boldsymbol{F}^{(\varphi)}(\varphi+\Delta\varphi)=([\sigma_{\ell\varphi}]_{\varphi+\Delta\varphi}[\boldsymbol{\varepsilon}_\ell]_{\varphi+\Delta\varphi}+[\sigma_{\varphi\varphi}]_{\varphi+\Delta\varphi}[\boldsymbol{\varepsilon}_\varphi]_{\varphi+\Delta\varphi}+[\sigma_{z\varphi}]_{\varphi+\Delta\varphi}\boldsymbol{\varepsilon}_z)\Delta\ell\Delta z \qquad (9)$$

Then, to evaluate the difference between $\vec{F}^{(\varphi)}(\varphi+\Delta\varphi)$ and $\vec{F}^{(\varphi)}(\varphi)$ we make use of the relationship

$$\frac{\partial\boldsymbol{a}_\ell}{\partial\varphi}=\left\{\begin{matrix}\alpha\\ \ell\,\varphi\end{matrix}\right\}\boldsymbol{a}_\alpha \quad , \quad \frac{\partial\boldsymbol{a}_\varphi}{\partial\varphi}=\left\{\begin{matrix}\alpha\\ \varphi\,\varphi\end{matrix}\right\}\boldsymbol{a}_\alpha$$

for the differentiation of the base vectors $\boldsymbol{a}_\ell$ and $\boldsymbol{a}_\varphi$ with respect to the parameter φ. Here, the quantity with three indices is the Christoffel's symbol of the second kind in tensor analysis [6], and the summation is implied for the repeated index $\alpha=\ell,\varphi,z$. It is found by using the relationships $\boldsymbol{a}_\ell=\boldsymbol{\varepsilon}_\ell$, $\boldsymbol{a}_\varphi=\ell\boldsymbol{\varepsilon}_\varphi$, $\boldsymbol{a}_z=\boldsymbol{\varepsilon}_z$ and the values of the Christoffel's symbol that these are explicitly given as

$$\frac{\partial\boldsymbol{\varepsilon}_\ell}{\partial\varphi}=\boldsymbol{\varepsilon}_\varphi \quad , \quad \frac{\partial\boldsymbol{\varepsilon}_\varphi}{\partial\varphi}=-\boldsymbol{\varepsilon}_\ell \qquad (10)$$

Utilizing this relationship (10), we obtain the difference $\delta\boldsymbol{F}^{(\varphi)}=\boldsymbol{F}^{(\varphi)}(\varphi+\Delta\varphi)-\boldsymbol{F}^{(\varphi)}(\varphi)$ as

$$\delta\boldsymbol{F}^{(\varphi)}=[\frac{\partial}{\partial\varphi}(\sigma_{\ell\varphi})\boldsymbol{\varepsilon}_\ell+(\sigma_{\ell\varphi})\boldsymbol{\varepsilon}_\varphi+\frac{\partial}{\partial\varphi}(\sigma_{\varphi\varphi})\boldsymbol{\varepsilon}_\varphi-(\sigma_{\varphi\varphi}\ell)\boldsymbol{\varepsilon}_\ell+\frac{\partial}{\partial\varphi}(\sigma_{z\varphi})\boldsymbol{\varepsilon}_z]\Delta\varphi\Delta\ell\Delta z \qquad (11)$$

It is then found that the force per unit volume $\boldsymbol{f}^{(\varphi)}=\delta\boldsymbol{F}^{(\varphi)}/\delta V$ is

$$\boldsymbol{f}^{(\varphi)}=\frac{1}{\ell}\frac{\partial}{\partial\varphi}(\sigma_{\ell\varphi})\boldsymbol{\varepsilon}_\ell+\frac{\sigma_{\ell\varphi}}{\ell}\boldsymbol{\varepsilon}_\varphi+\frac{1}{\ell}\frac{\partial}{\partial\varphi}(\sigma_{\varphi\varphi})\boldsymbol{\varepsilon}_\varphi-\frac{\sigma_{\varphi\varphi}}{\ell}\boldsymbol{\varepsilon}_\ell+\frac{1}{\ell}\frac{\partial}{\partial\varphi}(\sigma_{z\varphi})\boldsymbol{\varepsilon}_z \qquad (12)$$

Finally, a pair of the infinitesimal surfaces corresponding to the fixed values of z and $z+\Delta z$ is considered to obtain the force $\boldsymbol{f}^{(z)}$ per unit volume . Following the same procedure as for $\boldsymbol{f}^{(\ell)}$, we obtain

$$\boldsymbol{f}^{(z)}=\frac{\partial}{\partial z}(\sigma_{\ell z})\boldsymbol{\varepsilon}_\ell+\frac{\partial}{\partial z}(\sigma_{\varphi\varphi})\boldsymbol{\varepsilon}_\varphi+\frac{\partial}{\partial z}(\sigma_{zz})\boldsymbol{\varepsilon}_z \qquad (13)$$

The summation of these three forces $\boldsymbol{f}^{(\ell)}$, $\boldsymbol{f}^{(\varphi)}$, $\boldsymbol{f}^{(z)}$ is equal to the force $\boldsymbol{f}$ in equation (2) :

$$\boldsymbol{f}=\boldsymbol{f}^{(\ell)}+\boldsymbol{f}^{(\varphi)}+\boldsymbol{f}^{(z)} \qquad (14)$$

It is then straightforward to find the three components f_ℓ , f_φ , f_z in the direction of $\boldsymbol{\varepsilon}_\ell$, $\boldsymbol{\varepsilon}_\varphi$, $\boldsymbol{\varepsilon}_z$, respectively, from $\boldsymbol{f}$. The results are

$$f_\ell = \frac{1}{\ell}\frac{\partial}{\partial \ell}(\ell \sigma_{\ell\ell}) + \frac{1}{\ell}\frac{\partial}{\partial \theta}\sigma_{\ell\varphi} + \frac{\partial}{\partial z}\sigma_{\ell z} - \frac{1}{\ell}\sigma_{\varphi\varphi}$$

$$f_\varphi = \frac{1}{\ell}\frac{\partial}{\partial \ell}(\ell \sigma_{\varphi\ell}) + \frac{1}{\ell}\frac{\partial}{\partial \varphi}\sigma_{\varphi\varphi} + \frac{\partial}{\partial z}\sigma_{\varphi z} + \frac{1}{\ell}\sigma_{\ell\varphi} \tag{15}$$

$$f_z = \frac{1}{\ell}\frac{\partial}{\partial \ell}(\ell \sigma_{z\ell}) + \frac{1}{\ell}\frac{\partial}{\partial \varphi}\sigma_{z\varphi} + \frac{\partial}{\partial z}\sigma_{zz}$$

We consider the equilibrium state as a special case; the stress tensors in cartesian coordinates are diagonal and static ($\sigma_{ij} = -p\delta_{ij}$ with constant p). It is necessary that the force vanishes at equilibrium, as found from equation (2). By substituting $\sigma_{\alpha\beta} = -p\delta_{\alpha\beta}$ in equation (15), we easily find that the three components f_ℓ, f_φ, f_z vanish, or $\boldsymbol{f} = 0$. For f_ℓ, the first term is cancelled out by the last term. It is noted that the results given by equation (2.6.4) in [4] do not satisfy this requirement.

3. Force per unit volume in terms of space derivatives of the velocity

The velocity satisfies the continuity equation of the form

$$\nabla \cdot \boldsymbol{\upsilon} = 0 \tag{16}$$

in incompressible fluids. This gives the constraint for velocity components, whereby υ_ℓ, υ_φ, υ_z are not independent. In order to write equation (16) in cylindrical coordinates, we consider three pairs of the infinitesimal surface as mentioned above, and take the scalar products of υ with outer normal vectors by definition of divergence as $\boldsymbol{\varepsilon}_\ell \cdot \boldsymbol{\upsilon}\Delta S_\ell = \upsilon_\ell \ell \Delta\varphi\Delta z$, $\boldsymbol{\varepsilon}_\varphi \cdot \boldsymbol{\upsilon}\Delta S_\varphi = \upsilon_\varphi \Delta\ell\Delta z$, $\boldsymbol{\varepsilon}_z \cdot \boldsymbol{\upsilon}\Delta S_\ell = \upsilon_z \ell\Delta\ell\Delta\varphi$. It is then found by following the same procedure as shown to obtain $\boldsymbol{f}$ that $\nabla \cdot \boldsymbol{\upsilon}$ can be written with space derivatives of the velocity as

$$\nabla \cdot \boldsymbol{\upsilon} = \frac{1}{\ell}\frac{\partial}{\partial \ell}(\ell \upsilon_\ell) + \frac{1}{\ell}\frac{\partial \upsilon_\varphi}{\partial \varphi} + \frac{\partial \upsilon_z}{\partial z} \tag{17}$$

Substituting six components of the stress tensor [2] given by

$$\sigma_{\ell\ell} = -p + 2\eta\frac{\partial \upsilon_\ell}{\partial \ell},$$

$$\sigma_{\ell\varphi} = \sigma_{\varphi\ell} = \eta\left(\frac{\partial \upsilon_\varphi}{\partial \ell} + \frac{1}{\ell}\frac{\partial \upsilon_\ell}{\partial \varphi} - \frac{\upsilon_\varphi}{\ell}\right),$$

$$\sigma_{\ell z} = \eta\left(\frac{\partial \upsilon_z}{\partial \ell} + \frac{\partial \upsilon_\ell}{\partial z}\right) \tag{18}$$

$$\sigma_{\varphi\varphi} = -p + 2\eta\left(\frac{1}{\ell}\frac{\partial \upsilon_\varphi}{\partial \varphi} + \frac{\upsilon_\ell}{\ell}\right),$$

$$\sigma_{\varphi z} = \sigma_{z\varphi} = \eta\left(\frac{1}{\ell}\frac{\partial \upsilon_z}{\partial \varphi} + \frac{\partial \upsilon_\varphi}{\partial z}\right),$$

$$\sigma_{zz} = -p + 2\eta\frac{\partial \upsilon_z}{\partial z}$$

into equation (15) and using equation (16), we are able to show that the force per unit volume is written in terms of the space derivatives of the velocity as

$$f_\ell = -\frac{\partial p}{\partial \ell} + \eta\left[\left(\nabla^2 - \frac{1}{\ell^2}\right)\upsilon_\ell - \frac{2}{\ell^2}\frac{\partial \upsilon_\varphi}{\partial \varphi}\right],$$

$$f_\varphi = -\frac{1}{\ell}\frac{\partial p}{\partial \varphi} + \eta\left[\left(\nabla^2 - \frac{1}{\ell^2}\right)\upsilon_\varphi + \frac{2}{\ell^2}\frac{\partial \upsilon_\ell}{\partial \varphi}\right], \tag{19}$$

$$f_z = -\frac{\partial p}{\partial z} + \eta\nabla^2\upsilon_z,$$

with $$\nabla^2 = \frac{1}{\ell}\frac{\partial}{\partial \ell}\ell\frac{\partial}{\partial \ell} + \frac{1}{\ell^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

The equation (19) agrees with the results from a direct application of formal mathematical procedures to the vector Laplacian $\nabla^2\boldsymbol{\upsilon}$. It means that without the last term of $\sigma_{\ell\varphi}/\ell$ in f_φ in equation (15) one could not arrive at the expression of f_φ in equation (19).

In summary, we have discussed the physical significance of the net force in an infinitesimal volume, and obtained equation (15) as its expression in terms of stress tensors. The last terms $-\sigma_{\varphi\varphi}/\ell$ in f_ℓ, and $\sigma_{\ell\varphi}/\ell$ in f_φ in equation (15) are not involved in the Zamir's results of equation (2.6.4). It is because he just applied the procedure for the divergence of vectors in the cartesian coordinates, and did not take into consideration the so-called covariant derivative of vectors in curvilinear coordinates. We have pointed out the physical importance of these terms by considering its consequence in the case of static constant pressures. It would be interesting to investigate to what extent they are quantitatively important, compared to other terms, in dynamical problems.

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